Technical Note

PROOF, AND CORRECTION, OF SEVERE MISCONCEPTIONS, OF THE GENEAL ACCEPTED ANALYSIS OF STRESSES NEAR THE CRACK TIP

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ABSTRACT

It is shown, by following the applied derivation, that the general applied equations for stresses in the vicinity of the crack tip, are not right and have to be corrected, based on the ultimate stress condition of the tangential hydrostatic stress in the crack wall. The right equations show the indefinite high, linear-elastic, full hydrostatic failure stress, necessary to break molecular bonds at the crack tip and stresses therefore have to be given in the, first order, stress intensity form. Main error are:

- that the distance r, of the focus to the crack border(in fig. 2.2), which is of lower order for flat cracks, is regarded to be a free first order variable outside the crack border wherefore the equations do not apply.

- that the stress equations for stresses in solid material are applied, thus not on the crack border,

- that lower order small coordinate values and distances are compared with first order polar coordinate values. The correction leads to the mixed mode failure criterion, in first order stress intensities, of indefinitely high full-hydrostatic stresses.

.**Keywords:** Fracture mechanics, fracture limit analysis, notch boundary fracture analysis, isotropic and orthotropic materials, like wood, acting as a reinforced isotropic material.

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1. INTRODUCTION

In [1], the first, and still unique, exact analytical fracture mechanics derivation is given, of the critical stress state of an, in-plane, loaded plate with an elliptic hole in the middle, based on the solution of the strain differential equations in curvilinear elliptic coordinates. Applied is an always possible infinite series solution of differential equations, which did not lead to a new function, but, after satisfying general and boundary conditions, resulted in expressions in the form of existing transcendental functions (sin, cos, sinh, cosh, exp) which therefore represent the exact solution for the flat, thus sharp, crack problem. Determining for the strength is the boundary value solution of the determining tangential stress along the edge of the elliptic hole, which provides the mixed I-II mode fracture criterion. It appears that also Stevenson's [2] complex potentials solution of the Airy stress function, according to complex variable theory, discussed and applied, e.g. in [3], is based on this infinite series solution of Inglis [1], thus leading indirectly to an exact infinite series solution of the Airy stress function, because the same critical tangential stress equation along the crack border is obtained. In [3], the proof of this connection is given by the explicit derivation of the Griffith mixed I-II-mode failure criterion, by using the determining Stevenson's potentials, what leads to the same result as Griffith's 30 years older, Inglis-series based, solution. This Stevenson's potentials result appears to be copied by all textbooks, but is followed partially, thus incorrectly (see [4], [5]). Correction thus always is necessary. A fundamental extension is given for the critical state, e.g. in [4], [5], by the proof, that the exact equations in elliptical coordinates, show the full-hydrostatic critical stress state at the crack tip at any critical load combination. Thus for sharp cracks, the highest critical stress occurs at the crack tip and after curved shear line (slip line) formation, (as shown e.g. by digital image correlation technique), a full-hydrostatic stress field, occurs at the crack tip, instead of Irwin's plastic zone. It is shown in [4], that this high stress solution is a general property, necessary to break atomic bonds at fracture. The extension of this isotropic material fracture to fracture limit analysis of orthotropic materials like wood, (which thus behaves as reinforced material with an isotropic matrix) is e.g. given in [4]. It is also shown in [4], that the general applied fracture mechanics textbook crack tip boundary value stress problem solution of [6], is identical to the sum of the exact solutions of pure normal stress loading alone, and of pure shear stress loading alone, although these two solutions exclude each other and cannot apply at the same time. Evidently this, for stress and for displacements incompatible result is against the existing hydrostatic "mixed mode" failure criterion, which necessarily follows as solution of the crack boundary value problem. Further, by an improper small variables transformation, the Textbooks pure mode II solution is shown to be not correct. This all delivers, a wrong, incompatible, equilibrium system, which does not satisfy the crack boundary conditions and the failure criterion, thus should be omitted. As correction, the right exact limit analysis solution is given, e.g. in [4], which, as such, provides the derivation of "mixed I-II-mode" fracture criterion, which, as only equation, is verified, with sufficient precision, by empirical research. (see Table I, eq. 3). The necessary limit analysis approach for failure, as exact calculation method, provides the necessary linear elastic analysis up to the ultimate state (as occurs at reloading after unloading from the ultimate state). The tremendous high hydrostatic stress thus is a linear elastic stress far above the common plastic flow level. The fully empirical, so called, non-linear fracture mechanics, thus is a problem. Non linear processes in structural materials only can be described and analyzed by molecular deformation kinetics, thus with the non-linear chemical and physical reaction rate equations, (based on Bolzmann statistics) acting as non-linear dashpot and linear elastic spring constants of the applied Maxwell elements (see [7]). This is not possible with non-linear elasticity or deformation theory of plasticity. Stress is linear elastic, only and fully determined by linear elastic strain, thus by the distance between local material molecules.

2. PROOF OF GENERAL APPLIED DISASTROUS MISTAKES AT THE NEAR CRACK TIP, STRESS ANALYSIS

Because the same mathematical analysis is applied in all investigations, now and in the past, about the determining stresses near the crack tip, the catastrophic errors in this method are not noticed. One main mistake is, that the small scale elliptical coordinate variables and distances, which are several orders smaller than the crack length 2*a*, are wrongly compared with first order polar variables at the coordinate transition. This produces an additional apparent and incomplete first order load effect.
The exact mathematical fracture mechanics approach [3] and the more empirical [6] are generally referenced and followed since 1969, the year of the first prints of [3] and [6]. The exact equations, for the mixed mode strength determining tangential stress at the crack wall boundary, applied by Griffith for isotropic materials and in [8] for orthotropic materials, are known in elliptic coordinates, based on the infinite row solution of the displacements differential equations of Inglis, or by the related, Stevenson's [2] complex potentials solution of the Airy stress function. Because of failure, by the invariant full hydrostatic stress state, shown in [4] and [5], the equations are also directly known in polar and cartesian coordinates. This strength approach is not followed in Textbooks. Not strength relations are given, but only the finite declared, "infinite" high stresses near a, near zero, crack width of a, near zero flat, crack tip singularity. The description of the influence of small distances to the flat crack tip are regarded the best, by applying the confocal coordinate system of fig. 2.2. For the perfect flat crack the focus shifts to the crack tip, thus, in the limit, to a near zero distance. However, the derived stress equations are not applied to describe the ultimate higher order stress behavior near the crack tip, but are wrongly extrapolated to far distance stress equations, outside the crack border, and are thus unrelated, meaningless equations. The used mathematical equations only apply along the crack border. A main objection further is, that this, in fracture mechanics accepted, solution for combined stresses near the crack tip, only applies for stresses in solid material and is not right for the strength determining stress at the crack tip wall and thus cannot be applied in fracture mechanics. The proof of this, is given in the following, by comment on the followed derivation in [3].
The exact equations, based on the infinite series solution of the strain differential equations as basis for the stress potentials estimation of the Airy stress function solution, are in elliptical coordinates, what needs, (for applicability of the equations), transformation into Cartesian or polar coordinates. Therefore, first the relations between the elliptic and polar and cartesian coordinates are given, based on the confocal coordinate system of fig. 2.2, where the origin of the coordinate system is at a focus of the ellipse. Further, a to be corrected error is, that the, in the central coordinate system of Fig. 2.1, applied elliptic stress equations are based on application of the Maugis adaption of the Stevenson

potentials, for stresses in solid material, applied far outside the crack border. Thus they don't represent the determining stresses, which occur only along the boundary of the crack hole near the crack tip, as also is shown by the derivation of the exact mixed mode failure criterion in the following.
According to the derivation, given in [3] chapter 8.9, applies the following:
In Fig.2.1, the elliptic crack $\xi = \xi_o$ is given in a central (x,y) coordinate system. In Fig. 2.2, a related Cartesian (X,Y) and a polar (r,θ) coordinate systems is given, centered at a focal point.

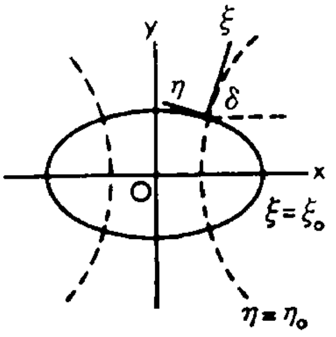

Fig. 2.1. Elliptic hole and elliptic coordinates
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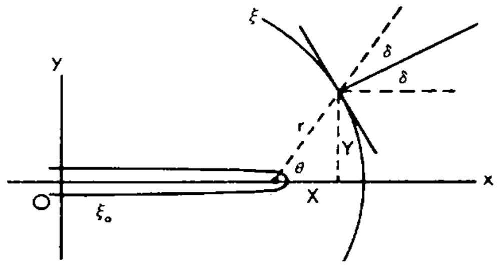

Fig. 2.2. Confocal polar coordinates for elliptic to polar coordinate transformation

For a point, in Fig. 2.2, near the elliptic crack tip of $\xi = \xi_o$, with coordinates:

$$x = c\cosh(\xi)\cos(\eta), \quad y = c\sinh(\xi)\sin(\eta), \tag{2.1}$$

applies, for small values of variable $z = \xi$ or η near the crack tip, that:
$\cosh(z) \approx 1+z^2/2$; $\cos(z) \approx 1- z^2/2$; $\sinh(z) \approx z$; $\sin(z) \approx z$,
as first expanded of a row expansions of the functions in z. Thus (2.1) becomes:

$$x = c(1 + \xi^2/2)(1- \eta^2/2) = c(1 + \xi^2/2 - \eta^2/2); \qquad y = c\,\xi\,\eta. \tag{2.1'}$$

With focus $x = c$, of a flat ellipse, as new origin for Cartesian coordinates X, Y, (see Fig. 2.2), applies, because of the small values of the elliptical coordinates ξ and η near the crack tip, that:

$$X = x - c = c\left(\xi^2 - \eta^2\right)/2, \quad Y = y = c\xi\eta \quad \rightarrow \tag{2.2}$$

$$r = \sqrt{X^2 + Y^2}, \quad X = r\cos(\theta), \quad Y = r\sin(\theta), \tag{2.3}$$

in Cartesian coordinates. And it follows that:

$$\xi^2 + \eta^2 = 2\left(X^2 + Y^2\right)^{1/2} = 2r/c \tag{2.4}$$

in elliptical coordinates. From (2.2) also follows:

$$X^2 = \left(\xi^4 - 2\xi^2\eta^2 + \eta^4\right)\frac{c^2}{4}; \;\; Y^2 = c^2\xi^2\eta^2; \;\; X^2 + Y^2 = \left(\xi^4 + 2\xi^2\eta^2 + \eta^4\right)\frac{c^2}{4} = \frac{c^2}{4}\left(\xi^2 + \eta^2\right)^2 = r^2$$

$$X^2 = \left(\xi^2 - \eta^2\right)^2\frac{c^2}{4} = \left(r\cos\theta\right)^2 \quad \rightarrow\left(\xi^2 + \eta^2\right) = \frac{2r}{c}; \quad \rightarrow\left(\xi^2 - \eta^2\right) = \frac{2r\cos(\theta)}{c}$$

Thus for the elliptic coordinates applies:

$$\xi=(r/c)^{1/2}(1+\cos(\theta))^{1/2}=(2r/c)^{1/2}\cos(\theta/2)=(\rho/c)^{1/2}\cos(\delta) \qquad \approx(\rho/c)^{1/2}(1-\delta^2/2) \quad (2.5)$$

$$\eta=(r/c)^{1/2}(1-\cos(\theta))^{1/2}=(2r/c)^{1/2}\sin(\theta/2)=(\rho/c)^{1/2}\sin(\delta) \qquad \approx(\rho/c)^{1/2}\delta \quad (2.6)$$

The real cyclic parameter, covering the whole range between 0 and π is not θ, but the surface curve normal value $\delta = \theta/2$ (see Fig. b) and the appurtenant radius is not r, but $\rho = 2r$, which is the radius of the crack tip (r is the distance of the nearest focus to the crack tip). A plot of the stress, dependent of ρ, should show every $\delta = \pi$ the same repeating plot, because the loading situation is identical every π degrees. This means that the Textbook stress equations give shifted values. When $\theta = \pi$ is inserted the stress value of $\delta = \pi/2$ is obtained. These shifted, incomplete values, are in the Textbooks and ***reformulation with ρ and δ, instead of r and θ, thus is needed.***

The angle δ, in fig. b, follows from:

$$e^{2i\delta}=\omega'(\zeta)/\bar{\omega}'(\zeta)=(\sinh(\xi+i\eta))/(\sinh(\xi-i\eta))=(\xi+i\eta)/(\xi-i\eta)=e^{i\theta} \quad (2.7)$$

or: $\delta = \theta/2$ (2.8)

The values ξ and η of eq.(2.5) and (2.6) are inserted in next eq.(2.9) to eq.(2.11), for stresses in solid material, near the crack tip, to get the required elliptical coordinate equations in polar coordinates. However this chosen insertion is not close to the crack tip, but applies everywhere in the field, because it is stated and regarded, that it applies at any value of θ and r.

***For values close to the crack tip, the first row expanded of sin(δ) and cos(δ) apply. Thus*:**

$\sin(\delta) \approx \delta \approx 0$, and $\cos(\delta) \approx 1 - \delta^2/2 \approx 1$, (2.1′′)

the same, as is applied for the functional related elliptical variables near the tip, given by eq.(2.1′).

Further, for the Textbook insertion,

wrongly both potentials are regarded, based on the Maugis modification (see [3]), which apply for solid material failure and not for fracture of loaded boundary walls of an open crack, thus don't apply for fracture mechanics crack extension. This will be corrected later. For the discussion of the consequences of this application, the applied Maugis equations have to be regarded in the following. This leads to:

$$\sigma_\xi+\sigma_\eta=p\cos(2\beta)+\alpha p\left[(1-\cos(2\beta))\sinh(2\xi)-\sin(2\beta)\sin(2\eta)\right] \quad (2.9)$$

$$\sigma_\xi-\sigma_\eta=\alpha p\cosh(2\xi)\cos(2(\eta-\beta))+\alpha^2 p\{(1-\cos(2\beta))(\cos(2\eta)-1)\sinh(2\xi)\}+$$
$$+\alpha^2 p\{-\cosh(2\xi)\cos(2\beta)+\cos(2(\eta-\beta))-\cosh(2\xi)\sin(2\beta)\sin(2\eta)\} \quad (2.10)$$

$$\tau_{\xi\eta}=\tfrac{1}{2}p\alpha\sinh(2\xi)\sin(2(\beta-\eta))+$$
$$+\tfrac{1}{2}p\alpha^2\{\sinh(2\xi)\sin(2\beta)(\cos(2\eta)-1)+(1-\cos(2\beta))(\cosh(2\xi)-1)\sin(2\eta)\} \quad (2.11)$$

with: $\alpha=(\cosh(2\xi)-\cos(2\eta))^{-1}= 0.5(\xi^2+\eta^2)^{-1}=c/4r$ (2.12)

Thus equations (2.9) to (2.11), in the central coordinate system (of Fig. 2.1), of a flat elliptical crack $\xi_0 = 0$, loaded by an uniaxial stress *p* at infinity inclined at β to the plane of the crack, followed directly by differentiation of both Stevenson potentials (see [3] for this and the equations).
The insert of ξ and η into eq.(2.9), for the flat crack $\xi_0 = 0$, for loading by stress *p* at infinity at an angle β to the crack, using small values properties of ξ and η, near the tip, gives:

$$\sigma_r + \sigma_\theta = \sigma_\xi + \sigma_\eta = p\cos(2\beta) + \alpha p\left[\left(1-\cos(2\beta)\right)\sinh(2\xi) - \sin(2\beta)\sin(2\eta)\right] =$$

$$= p\cos(2\beta) + \alpha p\left[\left(1-\cos(2\beta)\right)(2\xi) - (2\eta)\sin(2\beta)\right] =$$

$$= p\cos(2\beta) + p(c/2r)^{1/2}\left[\left(1-\cos(2\beta)\right)\cos(\theta/2) - \sin(2\beta)\sin(\theta/2)\right] \quad (2.9')$$

The first term, $p\cdot\cos(2\beta)$, in the last equation, is constant, but is negligible with respect to the term in $(r)^{-1/2}$. This means that $\sqrt{(c/r)}$ is of a higher order and thus *r* is two orders smaller than half the crack length *c*. This also follows from the coordinate values ξ and η which apply near the crack tip, by taking $\sinh(2\xi) \approx 2\xi$ and $\sin(2\eta) \approx 2\eta$, which next are regarded to be equal to: $\xi = (r/c)^{1/2}\cos(\theta/2)$ and $\eta = (r/c)^{1/2}\sin(\theta/2)$. Thus r/c is of lower order with respect to the lower order values ξ and η, which are, when being the lowest value in an equation, taken to be zero, ($\xi_0 = 0$, $\eta = 0$) thus is extremely small.
The stress equations apply only on the crack border: $\xi = \xi_0$. Thus the determining value of r is the distance of the focus to a crack tip border point. For sharp cracks this distance is of lower order and the focus is approximately on the crack tip (as is applied in [6]- A2).
This is not regarded in applications, and in Textbooks, where r is regarded to be a free variable and r/c, is always taken to be of order one, while in the derivation r is several orders smaller than half the crack length c. Regarded high r/c values thus represent points, far outside the applying mathematical equation, although they only apply at the crack border $\xi = \xi_0$. This delivers multiple order mistakes.
For the determining stress at the crack tip, applies, that $\eta = 0$ and $\theta = 0$, which are coordinate values of the crack tip. In the derivation also $\xi_o = 0$ is applied, so that then: $p\cdot\cos(2\beta)$ in eq.(p.9') indeed is several orders lower and has to be omitted from the equations (as always done).
This means that the crack is a closed slit, thus the crack width $b = 0$. Then also $\sqrt{(r_o)}$ in eq.(2.5): $x_o = (2r_o/c)^{1/2}$ has to be taken to be zero. This means that the always presented stress equations in appendix 2 of [6] and equations (8.246) to (8.275) of [3], all with r_o in the denominator, show not measurable undetermined high (thus hydrostatic!) stresses, which thus are all mutual equal, and thus are in the given form meaningless (infinite) and should be rewritten in stress intensities. Then is, for instance eq.(8.252) to eq.(8.254) of [3], for pure mode I loading, when $r \to 0$, and thus $\theta \to 0$:

$$\sigma_{rr} = \sigma_\perp\sqrt{c/2r}\cdot\cos(\theta/2)\left(1+\sin^2(\theta/2)\right) \approx \lim_{r\to 0}\left(\sigma_\perp\sqrt{c/2r}\right) \to \infty \quad [3]\ (8.252)$$

$$\sigma_{\theta\theta} = \sigma_\perp\sqrt{c/2r}\cdot\cos^3(\theta/2) \approx \lim_{r\to 0}\left(\sigma_\perp\sqrt{c/2r}\right) \to \infty \quad [3]\ (8.253)$$

$$\tau_{r\theta} = \sigma_\perp\sqrt{c/2r}\cdot\sin(\theta/2)\cos^2(\theta/2) \approx 0 \quad [3]\ (8.254)$$

The stress state thus is hydrostatic and is even triaxial full-hydrostatic because:

$\sigma_z = \nu\sigma_{rr} + \nu\sigma_{\theta\theta} = \sigma_{rr} = \sigma_{\theta\theta}$ because $\nu = 0.5$ of no volume change.

This agrees with the exact estimation in [4]. However, there remains yet one error in the equations:

By applying this analysis, based on both stress potentials, as done by Maugis, [3], (what applies for stresses in solid material), wrongly equal critical mode I and II stress intensities are obtained in all Textbooks.

The analysis thus is not right for failure near the crack tip and this only can be corrected by regarding the, potential for the fracture strength determining tangential stress, in the crack wall. The derivation of the mixed I-II mode failure equation, discussed in Section 3 below, shows that then the right mode I stress intensity for tension and right mode II stress intensity for shear stress loading are obtained in the equation.

Finally, after following the same Textbooks variable replacement procedure, as done for eq.(2.9) and solving the equations, it is found that:

$$\left(8r/(cp^2)\right)^{1/2}\sigma_r = \sin(\theta/2)\cdot\left(1-3\sin^2(\theta/2)\right)\sin(2\beta)+2\cos(\theta/2)\cdot\left(1+\sin^2(\theta/2)\right)\sin^2(\beta) \quad (2.14)$$

$$\left(8r/(cp^2)\right)^{1/2}\sigma_\theta = -3\sin(\theta/2)\cdot\cos^2(\theta/2)\cdot\sin(2\beta)+2\cos^2(\theta/2)\cdot\sin^2(\beta) \quad (2.15)$$

$$\left(8r/(cp^2)\right)^{1/2}\tau_{r\theta} = \cos(\theta/2)\cdot\left(3\cos^2(\theta/2)-2\right)\cdot\sin(2\beta)+2\cos^2(\theta/2)\cdot\sin(\theta/2)\cdot\sin^2(\beta) \quad (2.16)$$

which are the equations (8.246) to (8.248) of [3], of the elliptic to polar coordinate transformation. (applied in all Textbooks and thus also, literally present in [6] appendix 2 as shown by the derivation in [4], (eq.(6.29) to eq.(6.44)).

Differentiation of eq.(2.15) with respect to θ to find the strength determining value of the tangential stress σ_θ from $d(\sigma_\theta)/d\theta = 0$, delivers, that $\theta = \pi$ (trivial) or that θ follows from:

$$\tan(\beta)\sin(\theta)+3\cos(\theta)=1 \quad (2.17)$$

and it follows from eq.(2.16) that $\tau_{r\theta} = 0$ when eq.(2.17) is satisfied. Thus σ_θ and σ_r then are the principal stresses, of which σ_θ start as the highest.

For fracture near the crack tip: $\theta \to 0$, eq.(2.17), only applies when $\beta = \pi/2$, thus only for mode I, thus only for tension perpendicular to the crack direction and not for shear loading. Due to the high induced stress at the crack tip, the full hydrostatic stress state occurs, after flow of σ_r to σ_θ. The contraction coefficient $\nu = 0.5$ applies for both, plastic flow and full hydrostatic stress. Thus flow of the maximal stress, as mode I strength condition, delivers the full hydrostatic stress state after stress redistribution to 3 equal principal stresses, as applies due to the high peak stress due to the strong curvature (small *r*) at the tip of a flat crack. ***Thus Irwin's flow near the crack tip can be postulated to be equal to stress redistribution, up to the occurring full-hydrostatic stress state [4], with linear elastic principal stresses far above the plastic flow stress.***

The fact that the hydrostatic stress at the crack tip, only occurs for mode I in the Textbook equations, and not for shear loading, what is against the exact equations in elliptical coordinates, shows that the Textbook elliptical – polar coordinate transformations equations: (2.14) to (2.16), are not right in this respect. To control, whether the given mode I values, which are right at the crack tip, (by showing always the hydrostatic ultimate stress state), also are right, outside this crack tip region, the equations for pure mode I, thus for $\beta = \pi/2$, are regarded next. For that case, equations (2.14) to (2.16) become:

$$(2r/c)^{1/2}\sigma_r = p\cos(\theta/2)\cdot\left(1+\sin^2(\theta/2)\right) \tag{2.18}$$

$$(2r/c)^{1/2}\sigma_\theta = p\cos^3(\theta/2)\cdot \tag{2.19}$$

$$(2r/c)^{1/2}\tau_{r\theta} = p\cos^2(\theta/2)\sin(\theta/2) \tag{2.20}$$

In these equations is $r = \rho/2$, where ρ is the radius of the crack tip and $\delta = \theta/2$, the appurtenant elliptic rotation angle (see Fig. b). The principal stresses follow from:

$$\sqrt{c/2r}\cdot p\cos(\theta/2)\cdot\left(1\pm\sin(\theta/2)\right),\ \text{thus:}\ \sqrt{c/\rho}\cdot p\cdot\cos(\delta)\cdot\left(1\pm\sin(\delta)\right) \tag{2.21}$$

A plot of the mean principal stress value against δ, gives the mean load plot. According to eq.(2.21) this plot follows the form: $P\cdot\cos(\delta)$, due to the $\cos(\delta)$ term in the first order variable change eq.(2.5). Thus shows $+P$ at $\delta = 0$ and $-P$ at $\delta = \pi$, and next $+P$ at $\delta = 2\pi$. The repetition of the plot is at every: $\delta = 2\pi$, instead of the same load repetition every $\delta = \pi$, (as applies, in the right way, for the plot of pure mode II loading curve). The two principal stresses are equal at $\delta = 0$, showing the hydrostatic stress state at the crack tip. At increasing δ, one principal stress increases while the other decreases, up to $\delta \approx \pi/4$. This is the contribution of the opposite $\sin(\delta)$ terms in eq.(2.21). Next, both principal stresses decrease to zero at $\delta = \pi/2$, (in accordance with $\cos(\pi/2) = 0$). Thus then there is a zero total loading. Next, both become negative, one slowly and the other more strongly down to $\delta = 3\pi/4$. Then, one still decreases further, while and the other increases, down and up to $\delta = \pi$. Here both principal stresses are negative and equal and there thus is a negative hydrostatic stress state prediction. Then from $\delta = \pi$ to $\delta = 2\pi$, the same picture applies as going back from $\delta = \pi$ to $\delta = 0$. There thus is a precise asymmetric mean loading picture, between $\delta = 0$ and $\delta = \pi$, due to the cosines term in eq.(2.5), what is not right. There is no negative normal loading according to the boundary conditions and there should not be an antisymmetric point at $\delta = \pi/2$, with zero load. This point should be a symmetry point and at $\delta = \pi$, there should not be a negative hydrostatic stress but a positive one, the same as applies at $\delta = 0$. Clearly, the first order contributions: $\cos(\delta)$ and $\sin(\delta)$, in eq.(2.5) and eq.(2.6), have to be replaced by the lower order contributions: $\cos(\delta) \approx 1- \delta^2/2$ and: $\sin(\delta) \approx \delta$ as given by eq.(1''). This delivers the right hydrostatic stress state at the crack tip $\delta = 0$. Similar remarks apply for the mode II loading plot: For the, pure shear S, loading case of the transformed Textbook equations, follows from eq.(2.14) to (2.16), by the needed combination of two stresses of p = S at $\beta = \pi/4$ with p = - S at $\beta = 3\pi/4$, that:

$$(2r/c)^{1/2}\sigma_r = S\sin(\theta/2)\cdot\left(1-3\sin^2(\theta/2)\right)$$

$$(2r/c)^{1/2}\sigma_\theta = -3S\sin(\theta/2)\cos^2(\theta/2)$$

$$(2r/c)^{1/2}\tau_{r\theta} = S\cos(\theta/2)\cdot\left(3\cos^2(\theta/2)-2\right)$$

The principal stresses are given by:

$$\left(2r/cS^2\right)^{1/2}\sigma = -\sin(\theta/2) \pm 0.5\left(1+\cos^2(\theta)\right)$$

Here, at $\theta = 0$, the necessary occurring hydrostatic stress solution is shown to be disappeared.

Thus the Textbook coordinate transformation is not right for shear stress loading.

The fact, that also, at pure shear loading, the hydrostatic stress state has to occur at the crack tip, follows from the related exact equations in elliptical coordinates: eq.(9) to eq.(12) at the same loading, by stresses p at $\beta = \pi/4$ and - p at $\beta = 3\pi/4$. It then follows, that the stresses, in these equations, are:

$$\sigma_\xi = p\left(\cosh(2\xi)-1\right)\left(\alpha-\alpha^2\right)\sin(2\eta) \qquad \text{(p.22)}$$

$$\sigma_\eta = p\left[\alpha^2\left(\cosh(2\xi)-1\right)-\alpha\left(\cosh(2\xi)+1\right)\right]\sin(2\eta) \qquad \text{(p.23)}$$

$$\tau_{\xi\eta} = p\left[\alpha\cos(2\eta)-\alpha^2\left(1-\cos(2\eta)\right)\right]\sinh(2\xi) \qquad \text{(p.24)}$$

where $\alpha = 1/[\cosh(2\xi) - \cos(2\eta)]$. This delivers a hydrostatic stress state when: $\sigma_\xi = \sigma_\eta$, are equal principal stresses, and thus when, necessarily, also: $\tau_{\xi\eta} = 0$. For: $\sigma_\xi = \sigma_\eta$ is:

$$\left(\alpha-\alpha^2\right)\cosh(2\xi)-\alpha+\alpha^2 = -\left(\alpha-\alpha^2\right)\cosh(2\xi)-\alpha^2-\alpha \rightarrow$$

$$\rightarrow \cosh(2\xi) = \frac{\alpha}{\alpha-1} \approx 1+\frac{1}{\alpha} \approx 1+2\xi^2 \qquad \text{(p.25)}$$

and for $\tau_{\xi\eta} = 0$ is:

$$\alpha\cos(2\eta)-\alpha^2+\alpha^2\cos(2\eta) = 0 \quad \rightarrow \cos(2\eta) = \frac{\alpha}{\alpha+1} \approx 1-\frac{1}{\alpha} \approx 1-2\eta^2 \qquad \text{(p.26)}$$

Thus ξ and η are equal to $1/\sqrt{(2\alpha)} = \sqrt{(2r/c)}$, thus very small, close to zero for sharp cracks, because α is several orders higher than order one.

Substitution of eq.(p.25) and (p.26) into the appropriate stress equations (p.22) and (p.23), gives:

$$\sigma_\xi = \sigma_\eta = -p\alpha\sin(2\eta) = -p\alpha\frac{\sqrt{1+2\alpha}}{1+\alpha} \approx -p\sqrt{2\alpha} = -p\sqrt{\frac{c}{2r}} \qquad \text{(p.27)}$$

because $\alpha = (c/4r) >> 1$, where c is half the crack length. The third principal stress is:

$$\sigma_z = \upsilon\sigma_\xi + \upsilon\sigma_\eta = 0.5(\sigma_\xi+\sigma_\eta) = \sigma_\xi = \sigma_\eta \qquad \text{(p.28)}$$

Thus 3 equal stresses of full-hydrostatic behavior. The contraction coefficient is $v = 0.5$ of no volume change. In the limit this hydrostatic stress state at pure shear loading, occurs at the same place and magnitude as occurs at pure mode I loading. This therefore also applies for any load combination.

Applying the first row- expanded of: $\sin(\delta) \approx \delta \approx 0$ and $\cos(\delta) \approx 1 - \delta^2/2 \approx 1$, the equations become:

mode I, β = π/2 mode II, (β = π/4) + (β = - π/4)

$$\left(8r/(cp^2)\right)^{1/2}\sigma_r = 2\sin^2(\beta) = 2 \quad = \quad 1 \quad + \quad 1 \quad = \quad 2 \qquad (2.14')$$

$$\left(8r/(cp^2)\right)^{1/2}\sigma_\theta = 2\sin^2(\beta) = 2 \quad = \quad 1 \quad + \quad 1 \quad = \quad 2 \qquad (2.15')$$

$$\left(8r/(cp^2)\right)^{1/2}\tau_{r\theta} = \sin(2\beta) = 0 \quad = \quad 1 \quad + \quad -1 \quad = \quad 0 \qquad (2.16')$$

full-hydrostatic ↑ full-hydrostatic ↑

This delivers the hydrostatic stress state at the crack tip for pure tension (mode I) and pure shear, (mode II). Thus delivers the hydrostatic state for any load combination. Thus, only the second terms of equations (14) and (15), which are not zero at the top, when $\theta = 0$, count, and also the first term of eq.(16), which is not zero for $\theta = 0$. The equations thus become:

$\sigma_r = \sigma_\theta = \sigma_{hydr} \rightarrow \infty$ -- $\tau_{r\theta} = 0$ -- $K = p\sqrt{\pi c} = K_c = \sigma_{hydr}\sqrt{2\pi r}$ for: $\sigma_{hydr} \rightarrow \infty$ and $r \rightarrow 0$.

<u>Correction of the Textbook equations lead to a full hydrostatic critical stress state for any load combination.</u>

The small value coordinate relations of the semi-symmetric coordinate system of fig. 2.2 are inserted in the stress equations of the double symmetric coordinate system of Fig. 2.1, without applying the transformation equations for this change of coordinate axes, because this is not needed. Because, close to the crack tip, the hydrostatic stress state applies for fracture, no coordinate transformation equations are needed because they are already satisfied, because 2 principal stresses are equal. Thus the hydrostatic stress state is invariant.

<u>This result applies for the chosen case of solid material and does not apply for the determining open crack border $\xi = \xi_0$ for which the mathematical equations of fracture mechanics apply. Thus the here corrected equations, which still lead to a wrong mode I stress intensity, should not be applied in Textbooks.</u>

Of interest for a right result is only the exact mixed mode failure criterion in transformed hydrostatic stresses, discussed in the next section.

3. EXACT SOLUTION OF THE BOUNDARY VALUE PROBLEM OF THE "MIXED I-II MODE" FRACTURE CRITERION

The analysis of the boundary value problem of fracture mechanics of isotropic material is e.g. discussed in ([3], § 8.9-10). An extension for orthotropic materials, like wood and reinforced polymers, is derived in e.g. [8] and is discussed in [4]. The failure criterion depends on the actual ultimate full hydrostatic clear wood strength at the crack boundary, which is the highest loaded material near the crack tip. As mentioned, the Airy stress function solution of [2], which appears to be based on the exact solution of Inglis, is applied for estimation of the determining flat sharp crack stresses. Before the introduction of the fully empirical critical energy approach, which is based on non-existent mathematical and material properties, this was the general applied exact analysis of the past, which is still applied in rock mechanics [3], and for wood (orthotropic materials) [7]. To repeat the procedure: The mathematical solution of the Airy stress function equation:

$$\nabla^2\left(\nabla^2 U\right)=0 \tag{3.1}$$

is given in terms of two analytic functions ϕ(z) and χ(z), (where $z=x+iy$), which satisfy the conditions

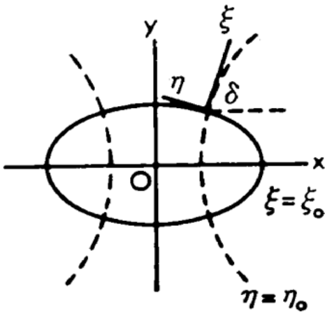

Figure 4.1. Elliptic hole and coordinates

at infinity and at the whole elliptic crack boundary surface, $\xi = \xi_o$, and show no discontinuity of displacements, thus represent the solution:

$$U = R\left\{\bar{z}\phi(z)+\chi(z)\right\}=0.5\left\{\bar{z}\phi(z)+z\bar{\phi}(z)+\chi(z)+\bar{\chi}(z)\right\} \tag{3.2}$$

The analytical functions (complex potentials) are based on the infinite row solution of Inglis, of the displacement differential equations. It appears, that all prescribed textbook equations: are based on the stress potential functions eq.(3.3) and eq.(3.4) which are given in [3].
For the elliptic hole $\xi = \xi_o$ with semi-axes: $a = c\cosh(\xi_o)$ and $b = c\sinh(\xi_o)$ in an infinite region loaded by an uniaxial stress p at infinity, inclined at β to the major axis Ox of the ellipse, (see Fig. 4.2), the functions ϕ(z) and χ'(z) = ψ(z) are, for the exact solution:

$$4\phi(z)=pce^{2\xi_0}\cos(2\beta)\cosh(\zeta)+pc\left(1-e^{2\xi_0+2i\beta}\right)\sinh(\zeta) \tag{3.3}$$

$$4\psi(z)=-pc[\cosh(2\xi_0)-\cos(2\beta)+e^{2\xi_0}\sinh\left(2\left(\zeta-\xi_0-i\beta\right)\right)]\cos ech(\zeta) \tag{3.4}$$

The tangential stress σ_t at the crack boundary $\xi = \xi_o$ is simply: $\sigma_t = \sigma_\eta$ because there: $\sigma_\xi = 0$, as boundary condition. Because the crack is empty there is no pressure and shear on the crack boundary surface. Thus $\sigma_\xi = 0$ and σ_t is:

$$\sigma_\xi+\sigma_\eta=2\left[\phi'(z)+\bar{\phi}'(z)\right]=\ \sigma_t=2\left[\phi'(\xi_0+i\eta)+\phi'(\xi_0-i\eta)\right]=$$

$$=pe^{2\xi_0}\cos(2\beta)+0.5p\left(1-e^{2\xi_0+2i\beta}\right)\coth(\xi_0+i\eta)+\ 0.5\,p\left(1-e^{2\xi_0-2i\beta}\right)\coth(\xi_0-i\eta)=$$

$$= pe^{2\xi_0}\cos(2\beta) + p\left[\cosh(2\xi_0) - \cos(2\eta)\right]^{-1}\sinh(2\xi_0) +$$

$$-pe^{2\xi_0}\left[\cos(2\beta)\sinh(2\xi_0) + \sin(2\beta)\sin(2\eta)\right]\left[\cosh(2\xi_0) - \cos(2\eta)\right]^{-1} =$$

$$= p\frac{\sinh(2\xi_0) + \cos(2\beta) - \exp(2\xi_0)\cos\left(2(\beta - \eta)\right)}{\cosh(2\xi_0) - \cos(2\eta)} \quad (3.5)$$

This eq.(3.5) can be extended by combining two stresses at infinity: p_2 inclined at β to Ox and p_1 at $\pi/2 + \beta$, making any loading combination (σ_y τ_{xy}) possible, according to:

$$\sigma_x = p_1\sin^2(\beta) + p_2\cos^2(\beta),\ \sigma_y = p_1\cos^2(\beta) + p_2\sin^2(\beta),\ \tau_{xy} = -0.5(p_1 - p_2)\sin(2\beta) \quad (3.6)$$

giving:

$$\sigma_t = \frac{2\sigma_y\sinh(2\xi_0) + 2\tau_{xy}[(1+\sinh(2\xi_0))\cot(2\beta) - \exp(2\xi_0)\cos\left(2(\beta-\eta)\right)\cos ec(2\beta)]}{\cosh(2\xi_0) - \cos(2\eta)} \quad (3.7)$$

For a flat, sharp crack, thus for small ξ_0 and η (near the crack tip) this is:

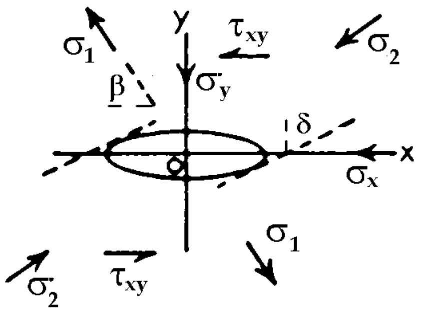

Figure 4.2. - Stresses in the notch plane Ox

$$\sigma_t = \frac{2\left(\xi_0\sigma_y - \eta\tau_{xy}\right)}{\xi_0^2 + \eta^2} \quad (3.8)$$

The determining maximal tangential stress follows from $d\sigma_t/d\eta = 0$. Thus:

$$\tau_{xy}\left(\xi_0^2 - \eta^2\right) + 2\xi_0\sigma_y\eta = 0 \quad \rightarrow \quad \eta = \xi_0\left[\sigma_y \pm \sqrt{\left(\sigma_y^2 + \tau_{xy}^2\right)}\right]/\tau_{xy}\ . \quad (3.9)$$

Substitutions in eq.(3.8) gives:

$$\xi_0\sigma_t = \sigma_y \pm \sqrt{\left(\sigma_y^2 + \tau_{xy}^2\right)} \quad (3.10)$$

This equation can be written:

$$\left(\xi_0\sigma_t - \sigma_y\right)^2 = \left(\pm\sqrt{\left(\sigma_y^2 + \tau_{xy}^2\right)}\right)^2 = \sigma_y^2 + \tau_{xy}^2, \quad \rightarrow$$

$$\frac{\tau_{xy}^2}{(\xi_0\sigma_t)^2} + \frac{\sigma_y}{\xi_0\sigma_t/2} = 1 \quad (3.11)$$

Transformation from elliptic to polar coordinates, by eq.(2.5):

$$\xi_0 = \sqrt{2r_0/c} = \sqrt{\rho/c} = b/c \quad (3.12)$$

where *b* and *c* are half axes lengths and substitution in eq.(3.11) gives:

$$\frac{\sigma_y\sqrt{\pi c}}{\sigma_t\sqrt{\pi r_0/2}}+\frac{\left(\tau_{xy}\sqrt{\pi c}\right)^2}{\left(\sigma_t\sqrt{2\pi r_0}\right)^2}=\frac{K_I}{K_{Ic}}+\frac{K_{II}^2}{\left(K_{IIc}\right)^2}=1 \qquad (3.13)$$

what is the mixed I-II mode fracture criterion [4], [8], for isotropic material.

Notice that no critical energy criterium is possible or required to obtain $K_I, K_{II}, K_{Ic}, K_{IIc}$ **.**

Eq.(3.11), with constant $\xi_o{\cdot}\sigma_t$, thus with always the same initial small crack length *c*, of clear wood, or of timber, gives and explains, the uniaxial, thus ultimate stress, strength criterion for wood. Strength theory and fracture mechanics are thus identical. In eq.(3.13) is the high value of σ_t, due to the small value of the crack tip radius r_o , thus due to the strong curvature. This delivers the critical stress intensity $\sigma_t\,(r_o)^{1/2}$ as ultimate strength parameter, with an unlimited high linear elastic, full-hydrostatic ultimate stress. Eq.(3.13). the 4th equation in Table I, is the only exact, (thus non-empirical), strength equation that therefore, cannot be rejected by the lack of fit test of Table I.

Eq.(3.13) replaces eq.(3.14), which is the 5th equation of Table I and is wrongly regarded to be fundamental e.g. in [6]-eq.2.36, but is rejected theoretically and empirically in Table I, as also applies for all other empirical equations.

$$G=\frac{K_I^2}{E'}+\frac{K_{II}^2}{E'}+\frac{K_{III}^2}{2\mu} \qquad (3.14)$$

Eq.(3.14) is based on a simple addition rule. Stated is, that the energy release rate, like energy, is a scalar quantity and thus can be added. Forgotten is however, the necessity of equilibrium, compatibility of stresses and strains and no discontinuity of displacements, etc. etc.

Table I. - Lack of fit values for supposed failure criteria [9]

Failure criterion	*p*-value
$K_I/K_{Ic}=1$	0.0001
$K_I/K_{Ic}+K_{II}/K_{IIc}=1$	0.0001
$K_I/K_{Ic}+\left(K_{II}/K_{IIc}\right)^2=1$	0.5629
$\left(K_I/K_{Ic}\right)^2+K_{II}/K_{IIc}=1$	0.0784
$\left(K_I/K_{Ic}\right)^2+\left(K_{II}/K_{IIc}\right)^2=1$	0.0001

4. CONCLUSIONS

- The underlined sentences represent the main conclusions about what is wrong with the near crack tip analysis in Textbooks and how it is corrected.
- The exact equations in elliptical coordinates, show, that at any single or combined mode I and II loading case, a full-hydrostatic ultimate stress state occurs at the crack tip.
- Fundamental also is, that in the following basic regarded Textbook equations:

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} F_{ij}(\theta) \quad \text{with: } K_I = \lim_{r \to 0} \sigma_{22} \cdot \sqrt{2\pi r}$$

the limit does not exist because the product: $\sigma_{22}\sqrt{(r)}$ is independent of r.

-- Because failure is always by the extremely high hydrostatic stresses, to break atomic bonds, this should be replaced by:

$$K_I = \sigma(\pi c)^{1/2} \le \lim_{\substack{r_o \to 0, \\ \sigma_{ij} \to \infty}} (\sigma_{ij} \cdot (\pi r_o / 2)^{1/2}) = K_{Ic} \quad \text{--} \quad K_{II} = \sigma(\pi c)^{1/2} \le \lim_{\substack{r_o \to 0, \\ \sigma_{ij} \to \infty}} (\sigma_{ij} \cdot (2\pi r_o)^{1/2}) = K_{IIc}$$

-- By applying the two Maugis potentials is wrongly $K_{Ic} = K_{IIc}$ obtained in the general accepted Textbook equations.

-- There should be no difference in strength theory and critical work theory.

-- Griffith did not intend to replace his strength theory by a critical energy theory. To hide the existence of an higher order linear elastic stress at a singularity (far above plane plastic flow stress), (which exists as full-hydrostatic stress), he reformulated his strength criterion into an apparent surface energy concept, what was a strength criterion expressed in the necessary amount of work per unit fractured surface formation. This thus is a constant material strength property, what later wrongly is called critical elastic strain energy release rate, although linear elastic strain energy is not involved in fracture itself but only provides displacement and stress compatibility with fracture movements. The visible fracture modes thus represent these linear elastic compatibility displacements.
According to Griffith there thus is no difference between strength theory and right energy approach.

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